

Universal weight systems and the Melvin-Morton expansion of the colored Jones knot invariant

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Abstract

We study the asymptotic expansion of the colored Jones polynomial (the Melvin-Morton expansion) using a recursive formula for the deframed universal weight system for the sl_2 Lie algebra. Combined with the formula for the universal weight system for the Lie superalgebra $gl(1|1)$ (which corresponds to the Alexander knot polynomial) this formula gives a very short proof of the Melvin-Morton conjecture relating the colored Jones invariant and the Alexander polynomial of knots.

1 Introduction

The two most famous invariants of oriented links in S^3 , the Alexander and Jones polynomials, have very similar combinatorial definitions.

Let U be the unknot and K_+ , K_- , and K_0 three knots (or links) identical outside of a small ball whose intersection with this ball looks as shown on Fig. 1.

The Alexander polynomial Δ is the link invariant uniquely determined by the following conditions (skein relations)

$$\Delta(K_+, t) - \Delta(K_-, t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(K_0, t), \quad \Delta(U, t) = 1, \quad (1)$$

and the Jones polynomial of links is defined by the relations

$$t^{-1}V(K_+, t) - tV(K_-, t) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V(K_0, t), \quad V(U, t) = 1. \quad (2)$$

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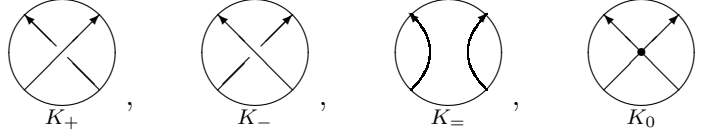


Figure 1:

Both Δ and V are Laurent polynomials in \sqrt{t} , but if K has only one component (i.e. K is a knot), $\Delta(K)$ and $V(K)$ belong to $\mathbf{C}[t, t^{-1}]$.

Despite the similarity between the skein relations for the Jones and Alexander invariants, there is a huge gap in our understanding of their nature. The Alexander polynomial has a solid topological meaning (as a determinantal invariant of the $\mathbf{Z}[t, t^{-1}]$ -module $H^1(\widetilde{M})$, where $\widetilde{M} \rightarrow S^3 \setminus K$ is the infinite cyclic cover of the knot complement corresponding to the commutator subgroup $G' = [G, G]$ of the knot group $G = \pi_1(S^3 \setminus K)$).

The Jones polynomial, on the other hand, is defined purely combinatorially and even though it was used to settle some very old conjectures in knot theory, its topological meaning is still very obscure.

The theory of quantum groups offered a new viewpoint on the Jones polynomial. It was found that $V(K)$ can be constructed by working with the standard two-dimensional representation of the quantum group $sl_{2,q}$. An arbitrary, d -dimensional, representation of $sl_{2,q}$ gives rise to a generalization of V , the so-called *colored Jones invariant* V^d (see [10, 16] for more details).

Morton and Strickland [14] found a combinatorial formula for computing $V^d(K, t)$ proving that V^d is determined by all the cablings of the Jones polynomial.

In [12] Melvin and Morton studied pieces of $V^d(K, t)$ appearing in a certain power series expansions and conjectured that the Alexander polynomial of a knot is determined by its colored Jones invariant. They proved that the coefficients v_{in} of the power series expansion

$$V^d(K, e^z) = \sum_{i,n \geq 0} v_{in}(K) d^i z^n \quad (3)$$

of V^d in variables d and $z = \log t$ are Vassiliev knot invariants of order $\leq n$ and that $v_{in} = 0$ if $i > 2n$. They formulated the following *Melvin-Morton conjecture* (later Morton [13] proved it for torus knots).

Theorem 1.1

(i) The coefficient matrix (v_{in}) of the expansion (3) is “lower triangular,” i.e.

$$v_{in} = 0 \quad \text{for } i > n \quad (4)$$

(ii) the leading (“diagonal”) term

$$V_0(K, z) = \sum_{n \geq 0} v_{nn}(K) z^n \quad (5)$$

is the inverse of the renormalized Alexander polynomial of K

$$V_0(K, z) \cdot \frac{z}{e^{z/2} - e^{-z/2}} \Delta(K, e^z) = 1 . \quad (6)$$

Rozansky [17] derived relations (4) and (6) from Witten’s [22] path integral interpretation of the Jones invariant using quantum field theory tools.

Finally, Bar-Natan and Garoufalidis [4] proved the Melvin-Morton conjecture by finding nice combinatorial expressions for the weight systems corresponding to the leading terms v_{nn} of the expansion (3) and the coefficients of the Alexander polynomial. Their method cannot be used to study off-diagonal coefficients of (3).

One of the difficulties in studying the colored Jones invariant is that it deals with all representations of sl_2 at once. In this paper we suggest a different approach to analyzing the Melvin-Morton expansion (3) for which this problem does not arise. Our method is based on universal Vassiliev invariants — Vassiliev knot invariants corresponding to weight systems with values in the center $Z(\mathcal{U}(L))$ of the universal enveloping algebra of a Lie (super)algebra L . The sl_2 universal Vassiliev invariant is a power series with coefficients in $Z(\mathcal{U}(sl_2)) = \mathbf{C}[c]$, where c is the quadratic Casimir. It is equivalent to the colored Jones invariant, since in the d -dimensional representation c acts as the scalar $(d^2 - 1)/2$. Thus, we can study the colored Jones invariant entirely in terms of sl_2 alone without working with its different representations.

Very little is known about universal Vassiliev invariants and the corresponding weight systems in general. The only non-trivial examples with complete answers are the cases of the Lie algebra sl_2 and of the Lie superalgebra $gl(1|1)$ which correspond respectively to the colored Jones and the Alexander knot invariants.

Chmutov and Varchenko [5] found a recursion formula for the universal sl_2 invariant for *framed* knots. In this paper we derive a similar relation for the sl_2 invariant of *unframed* knots (i.e. of the colored Jones invariant) that enables us to compute recursively the weight systems corresponding to all the “lines”

$$V_p = \sum_{n \geq 0} v_{n, n+p} z^n$$

of the expansion (3).

The Alexander polynomial is, essentially, the universal invariant for the simplest interesting Lie superalgebra $gl(1|1)$ (cf. [8, 20, 19, 6]). In [6] we studied the universal $gl(1|1)$ invariants (both for framed and unframed knots) and found recursion formulas for the corresponding weight systems.

The recursive relation for the deframed sl_2 -invariant looks very similar to the one for $gl(1|1)$. This gives a very simple proof of Theorem 1.1 and paves a way to a better understanding the lower lines V_p in (3).

The proofs of the recursion formulas for both the sl_2 and $gl(1|1)$ universal invariants are based on two fundamental relations (24) and (41) between invariant tensors of order four on these Lie (super)algebras.

The relation for sl_2 has been known for a long time. In the so_3 realization of sl_2 it is the famous Lagrange's identity of vector algebra in \mathbf{R}^3 :

$$[\mathbf{a} \times \mathbf{b}] \cdot [\mathbf{c} \times \mathbf{d}] = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) .$$

The corresponding relation for $gl(1|1)$ was discovered in [6].

These relations, in fact, are responsible for the skein relations (1) and (2) and for other properties of the Jones and Alexander knot invariants.

The plan of the paper is as follows. In Section 2 we collect necessary information on Vassiliev knot invariants and their relations with Lie algebra-type structures. In particular, we explain, following [4], why it is enough to establish (4) and (6) on the level of weight systems. In Section 3 we discuss the weight systems for the coefficients of the Melvin-Morton expansion (3) of the colored Jones invariant. In particular, we prove a recursion formula for the deframed universal sl_2 weight system. In Section 4 we review the results of [6] on the universal $gl(1|1)$ Vassiliev invariant and its relation with the Alexander polynomial. Then we show that Theorem 1.1 is a simple corollary of our formulas for the universal sl_2 and $gl(1|1)$ invariants.

2 Vassiliev invariants and universal weight systems

Here we review some facts about Vassiliev invariants and their relationship with Lie algebra-type structures. For more details see [3, 9, 19].

2.1 Vassiliev invariants, chord diagrams and weight systems

A *singular knot* is an immersion $K : S^1 \rightarrow \mathbf{R}^3$ with a finite number of double self-intersections with distinct tangents. The set of singular knots with n double points is denoted by \mathcal{K}_n .

A *chord diagram* of order n is an oriented circle with n disjoint pairs of points (*chords*) on it up to an orientation preserving diffeomorphism of the circle. Denote by \mathcal{D}_n the set of all chord diagrams with n chords.

Every singular knot $K \in \mathcal{K}_n$ has a chord diagram $ch(K) \in \mathcal{D}_n$ whose chords are the inverse images of the double points of K .

Every knot invariant I with values in an abelian group k extends to an invariant of singular knots by the rule

$$I(K_0) = I(K_+) - I(K_-), \quad (7)$$

where K_0 , K_+ , and K_- are singular knots which differ only inside a small ball as shown on Fig. 1.

A knot invariant I is called an *invariant of order* (\leq) n if $I(K) = 0$ for any $K \in \mathcal{K}_{n+1}$.

The k -module of all invariants of order n is denoted by \mathcal{V}_n . We have the obvious filtration

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \mathcal{V}_2 \dots \subset \mathcal{V}_n \subset \dots$$

Elements of $\mathcal{V}_\infty = \bigcup_n \mathcal{V}_n$ are called *invariants of finite type* or *Vassiliev invariants*.

Similarly the space \mathcal{V}_n^f of Vassiliev invariants of framed knots is defined.

An immediate corollary of the definition of Vassiliev invariants is that the value of an invariant $I \in \mathcal{V}_n$ on a singular knot K with n self-intersections depends only on the diagram $ch(K)$ of K . In other words, I descends to a function on \mathcal{D}_n which we still denote by I . These functions satisfy two groups of relations

$$I \left(\text{diagram of a circle with a vertical chord} \right) = 0 \quad (8)$$

and

$$I \left(\text{diagram 1} \right) - I \left(\text{diagram 2} \right) + I \left(\text{diagram 3} \right) - I \left(\text{diagram 4} \right) = 0. \quad (9)$$

Besides the explicitly shown chords, the diagrams may contain other chords with endpoints lying on the dotted arcs, provided that these chords are the same in all diagrams appearing in the same relation.

A function $W : \mathcal{D}_n \rightarrow k$ is called a *weight system of order n* if it satisfies the *four-term relations* (9). If, in addition, W satisfies the *one-term relations* (8), we call it a *strong weight system*.

Denote by \mathcal{W}_n (resp. by $\overline{\mathcal{W}}_n$) the set of all weight systems (resp. strong weight systems) of order n .

A Vassiliev invariant (resp. a Vassiliev invariant of framed knots) of order n defines a strong weight system (resp. a weight system) of order n and it is easy to see that the natural maps $\mathcal{V}_n^f / \mathcal{V}_{n-1}^f \rightarrow \mathcal{W}_n$ and $\mathcal{V}_n / \mathcal{V}_{n-1} \rightarrow \overline{\mathcal{W}}_n$ are injective.

The remarkable fact proved by Kontsevich [9] is that these maps are also surjective (at least when $k \supset \mathbf{Q}$). In other words, each (strong) weight system of order n is a restriction to \mathcal{D}_n of some Vassiliev invariant.

Kontsevich proved that

$$\mathcal{V}_n/\mathcal{V}_{n-1} \simeq \overline{\mathcal{W}}_n, \text{ and } \mathcal{V}_n^f/\mathcal{V}_{n-1}^f \simeq \mathcal{W}_n$$

by explicitly constructing splitting maps

$$Z : \mathcal{W}_n \rightarrow \mathcal{V}_n^f \quad \text{and} \quad \bar{Z} : \overline{\mathcal{W}}_n \rightarrow \mathcal{V}_n \quad (10)$$

If k is a commutative ring, then the product of two Vassiliev invariants of orders m and n is a Vassiliev invariant of order $m+n$, therefore \mathcal{V} is a filtered algebra.

The space

$$\bigoplus_{n \geq 0} \mathcal{W}_n = \bigoplus_{n \geq 0} \mathcal{V}_{n+1}^f / \mathcal{V}_n^f$$

becomes the adjoint graded algebra of \mathcal{V} with the product defined as follows. For $W \in \mathcal{W}_m$, $W' \in \mathcal{W}_n$ and $D \in \mathcal{D}_{m+n}$ we have

$$(W \cdot W')(D) = \sum_{\substack{E \subset D, \\ |E|=m}} W(E)W'(D \setminus E). \quad (11)$$

Many knot invariants such as the Alexander and Jones polynomials are not Vassiliev invariants, whereas their coefficients (after an appropriate change of variables) are. Therefore, we can associate with such invariants not just one weight system, but a sequence of weight systems $w_0, w_1, \dots, w_n, \dots$ where $w_n \in \mathcal{W}_n$. We call elements of $\mathcal{W} = \prod_{n \geq 0} \mathcal{W}_n$ *Vassiliev series*

and write them as formal sums $W = w_0 + w_1 + w_2 + \dots$ (or sometimes as formal power series $\sum w_n z^n$, where z is a formal parameter). Vassiliev series can be viewed as linear functionals on the space of diagrams $\mathcal{D} = \bigoplus_{n \geq 0} \mathcal{D}_n$: for $D \in \mathcal{D}_n$ and $W \in \mathcal{W}$ we define $W(D) = w_n(D)$.

The space \mathcal{W} also has an algebra structure

$$(W \cdot W')(D) = \sum_{E \subset D} W(E)W'(D \setminus E). \quad (12)$$

Every weight system of order n gives a strong weight system of order n by means of a canonical projection (*deframing*)

$$p : \mathcal{W}_n \rightarrow \overline{\mathcal{W}}_n \quad (13)$$

given by

$$p(W)(D) = \sum_{E \subset D} (-1)^{|E|} W(\Theta^{|E|} \cdot (D \setminus E)), \quad (14)$$

where Θ is the unique chord diagram with one chord and the product of two chord diagrams D_1 and D_2 is just their connected sum $D_1 \cdot D_2$. The product of diagrams is well-defined modulo four-term relations (9).

A Vassiliev series $W \in \mathcal{W}$ is called *multiplicative* if

$$W(D_1 \cdot D_2) = W(D_1) \cdot W(D_2) \text{ for any } D_1, D_2 \in \mathcal{D}.$$

Combining equations (12) and (14) we obtain a convenient formula for deframing multiplicative weight systems.

Proposition 2.1 *Let W be a multiplicative Vassiliev series and U be the Vassiliev series*

$$U(D) = (-c)^{|D|}, \text{ where } c = W(\Theta).$$

Then the deframed strong weight system $\overline{W} = p(W)$ is given by

$$\overline{W} = W \cdot U \quad \text{or} \quad \overline{W}(D) = \sum_{E \subset D} (-c)^{|E|} \cdot W(D \setminus E). \quad (15)$$

□

2.2 Weight systems coming from Lie algebras

Here we recall a construction that assigns a family of weight systems to every Lie (super)algebra with an invariant inner product.

First, we introduce a more general class of diagrams.

A *Feynman diagram* of order p is a graph with $2p$ vertices of degrees 1 or 3 with a cyclic ordering on the set of its univalent (*external*) vertices and on each set of three edges meeting at a trivalent (*internal*) vertex.¹ Let \mathcal{F}_p denote the set of all Feynman diagrams with $2p$ vertices (up to the natural equivalence of graphs with orientations). The set \mathcal{D}_p of chord diagrams with p chords is a subset of \mathcal{F}_p .

Any weight system $W : \mathcal{D}_p \rightarrow k$ can be extended to \mathcal{F}_p by the following rule, the *three-term relations*

$$W \left(\text{Diagram } D_Y \right) = W \left(\text{Diagram } D_{||} \right) - W \left(\text{Diagram } D_X \right). \quad (16)$$

The space $\langle \mathcal{F}_p \rangle / \langle D_Y - D_{||} + D_X \rangle$ of Feynman diagrams modulo three-term relations is isomorphic to the space generated by \mathcal{D}_p modulo four-term relations (9), i.e. it is canonically dual to \mathcal{W}_p .

¹Feynman diagrams are called Chinese Character diagrams in [3], but they are indeed Feynman diagrams arising in the perturbative Chern-Simons-Witten quantum field theory [22, 2].

In addition, the following local relations hold for internal vertices of Feynman diagrams

$$W \text{ (figure-eight) } = -W \text{ (cup) } \quad \text{and} \quad W \text{ (crossing) } = W \text{ (Y-junction) } - W \text{ (X-junction)}. \quad (17)$$

we have

$$T_L(C) = \sum_{ij} b^{ij} e_i \otimes e_j = c,$$

the Casimir element corresponding to the inner product b ,

$$T_L(B) = \sum b^{is} b^{tj} b^{kp} b^{lq} f_{skl} f_{pqt} e_i \otimes e_j,$$

the tensor in $L \otimes L$ corresponding to the Killing form on L under the identification $L^* \simeq L$, and

$$T_L(K) = \sum b^{in} b^{jp} b^{qr} b^{kt} b^{\ell s} f_{npq} f_{tsr}^k e_i \otimes e_j \otimes e_k \otimes e_\ell,$$

where f_{jk}^i are the structure constants of L in a basis e_1, e_2, \dots .

Tensor $T_L(F)$ is invariant with respect to the L -action on $L^{\otimes m}$ and its image $W_L(F)$ in the universal enveloping algebra $\mathcal{U}(L)$ belongs to the center $Z(\mathcal{U}(L)) = \mathcal{U}(L)^L$ and does not depend on the place where we cut the Wilson line to obtain a linear ordering of the external vertices of F .

The conditions (17) and (16) are automatically satisfied for W_L : the relations (17) are the anticommutativity and the Jacobi identity for the Lie bracket, and (16) is just the definition of the universal enveloping algebra as a quotient of the tensor algebra of L .

Therefore, for every Lie algebra L with an invariant inner product there exists a natural Vassiliev series $W_L : \mathcal{D} \rightarrow Z(\mathcal{U}(L))$.

The Vassiliev series

$$W_L : \mathcal{D} \rightarrow Z(\mathcal{U}(L))$$

is called *the universal weight system* corresponding to L (we are suppressing b from the notation). It is universal in the sense that any Vassiliev series $W_{L,R}$ constructed using a representation R of the Lie algebra L (see [3]) is an evaluation of W_L :

$$W_{L,R}(D) = \text{Tr}_R(W_L(D)).$$

By its construction, the universal Vassiliev series W_L is multiplicative, i.e. $W_L(D_1 \cdot D_2) = W_L(D_1)W_L(D_2)$.

2.3 Reduction of the Melvin-Morton conjecture to weight systems

Given a semi-simple Lie algebra L with an invariant inner product b and a representation R , there are two ways to construct knot invariants from this data. First, we can use the Reshetikhin-Turaev construction [16] based on quantum groups to get invariants $I_{L,R}$ and $\bar{I}_{L,R}$ (of framed and unframed knots, resp.) with values in $\mathbf{Z}[t, t^{-1}]$. Second, we can apply Kontsevich's splitting map to Vassiliev series $W_{L,R} = \text{Tr}_R(W_L)$ and $\bar{W}_{L,R}$ discussed in the previous section.

The remarkable fact is that these two constructions are equivalent. As it was first noticed by Birman and Lin [1], the coefficients of the power series expansions of $\bar{J}_{L,R}(z) = \bar{I}_{L,R}(e^z)$ are Vassiliev invariants. Piunikhin [15] proved that the corresponding series of weight systems coincide with $\bar{W}_{L,R}$. Kassel [7] and Le and Murakami [11] showed that, conversely, Kontsevich's construction applied to $\bar{W}_{L,R}$ (resp. to $W_{L,R}$) gives the sequence of the coefficients of $\bar{J}_{L,R}$ (resp. $J_{L,R}$).

A Vassiliev invariant is called *canonical* if it belongs to the image of Kontsevich's map $Z : \mathcal{W}_n \rightarrow \mathcal{V}_n$. A formal power series $\sum_{n \geq 0} v_n z^n \in \mathcal{V}[[z]]$ is called *canonical* if every coefficient v_n is a canonical Vassiliev invariant of order $\leq n$.

The colored Jones invariant is the canonical invariant $Z(\frac{1}{d}\bar{W}_{sl_2, R_d})$ corresponding to the d -dimensional representation of sl_2 . Since a canonical invariant is uniquely determined by its weight system, to prove part (i) of the Melvin-Morton conjecture (1.1) it is enough to show that the weight system of v_{in} in (3) vanishes for $i > n$.

Bar-Natan and Garoufalidis [4] proved that

$$\tilde{\Delta}(z) = \frac{z}{e^{z/2} - e^{-z/2}} \Delta(e^z), \quad (19)$$

where $\Delta(K, t)$ is the Alexander polynomial of knot K , is a canonical series. (This also follows from the generalization of the Kassel-Le-Murakami theorem for classical Lie superalgebras and from the fact that the Alexander invariant and the corresponding Vassiliev series C come from $gl(1|1)$, see [8, 20, 6]).

The product of two canonical invariants or Vassiliev series $Z(W_1)$ and $Z(W_2)$ is again canonical with the weight system equal to $W_1 W_2$. Therefore, to prove part (ii) of the Melvin-Morton conjecture it is enough to establish that the Vassiliev series corresponding to the two factors in (6) are inverses of each other.

Theorem 1.1 is now reduced to the following relations on the level of weight systems.

Proposition 2.2 *Let*

$$\widehat{W}_{sl} = \frac{1}{d} \sum_{n \geq 0} \bar{W}_{sl_2, R_d, n} = \sum_{i, n \geq 0} w_{in} d^i \quad (20)$$

be the deframed Vassiliev series (weight system) coming from the d -dimensional representation of sl_2 with the standard metric $\langle x, y \rangle = \text{Tr}(xy)$ (normalized by dividing by d). Then

$$(i) \quad w_{in} = 0 \quad \text{for } i > n \quad (21)$$

and

$$(ii) \quad \left(\sum_{n \geq 0} w_{nn} \right) \cdot C = \varepsilon, \quad (22)$$

where $C = \sum C_n z^n \in \mathcal{W}$ is the Vassiliev series of the normalized Alexander invariant $\tilde{\Delta}$ (19) and ε is the Vassiliev series

$$\varepsilon(D) = \begin{cases} 1 & \text{if } |D| = 0, \\ 0 & \text{if } |D| > 0. \end{cases}$$

Bar-Natan and Garoufalidis [4] proved (22) by finding combinatorial formulas for both weight systems in the left-hand side of this equation. We will show that both (21) and (22) are simple corollaries of the recursion formulas for the sl_2 and $gl(1|1)$ universal weight systems.

3 Universal sl_2 weight systems

The standard choice of an invariant metric on $L = sl_2$ is

$$\langle x, y \rangle = \text{Tr}_V(xy).$$

This metric is equal to one fourth of the Killing form $\langle x, y \rangle_K = \text{Tr}(ad_x ad_y)$, therefore

$$W_L \text{ (diagram with a circle on a chord) } = 4W_L \text{ (diagram with a single arc) } . \quad (23)$$

The center of $\mathcal{U}(L)$ is $\mathbf{C}[c]$ where c is the quadratic Casimir — the element of $S^2(L)$ corresponding to the metric under the identification $L^* \simeq L$.

In this setting

$$W(D) = c^n - 2pc^{n-1} + \text{terms of lower order in } c,$$

where $n = |D|$ is the number of chords in the diagram D , and p is the number of pairs of intersecting chords in D .

The following relation between invariant tensors in $L^{\otimes 4}$ is responsible for all properties of the Vassiliev series W_L .

Let

$$P = \text{(diagram with two intersecting arcs)} - \text{(diagram with two nested arcs)}$$

and K be the Feynman diagram in (18). Then

$$W_L(K) = -2W_L(P),$$

In other words

$$W_L \text{ (diagram with two intersecting arcs) } = 2W_L \text{ (diagram with two nested arcs) } - 2W_L \text{ (diagram with two intersecting arcs) } , \quad (24)$$

i.e.

$$\langle [a, b], [c, d] \rangle = 2\langle a, d \rangle \cdot \langle b, c \rangle - 2\langle a, c \rangle \langle b, d \rangle \text{ for } a, b, c, d \in sl_2. \quad (25)$$

Under the isomorphism $sl_2 \simeq so_3$, the inner product and the Lie bracket on sl_2 become respectively the scalar and the vector product in the three-dimensional space and (25) becomes the classical Lagrange's identity²

$$[\mathbf{a} \times \mathbf{b}] \cdot [\mathbf{c} \times \mathbf{d}] = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

which is equivalent to the better known *fundamental relation of vector calculus*

$$[\mathbf{a} \times [\mathbf{b} \times \mathbf{c}]] = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}.$$

Identity (24) allows us to compute the universal sl_2 framed and unframed weight systems recursively.

Theorem 3.1 *Let D be a chord diagram, “ a ” a fixed chord in D , and let $I_a = \{b_1, b_2, \dots, b_p\}$ be the set of chords in D intersecting a . Denote by D_a (resp. $D_{a,i}$, $D_{a,ij}$) the diagram $D - a$ (resp. $D - a - b_i$, $D - a - b_i - b_j$). Then the framed W and unframed \overline{W} universal sl_2 weight systems satisfy the following recursion relations*

$$(i) \quad W(D) = (c - 2p)W(D_a) + 2 \sum_{i < j} \left(W(D_{a,ij}^{\parallel}) - W(D_{a,ij}^{\times}) \right), \quad (26)$$

and

$$(ii) \quad \overline{W}(D) = -2|I_a|\overline{W}(D_a) - 2c \sum_{i \in I_a} \overline{W}(D_{a,i}) + 2 \sum_{i < j} \left(\overline{W}(D_{a,ij}^{\parallel}) - \overline{W}(D_{a,ij}^{\times}) \right) \\ - 2c \sum_{i < j} \left(\overline{W}(D_{a,ij}^{lr}) + \overline{W}(D_{a,ij}^{rl}) - \overline{W}(D_{a,ij}^{ll}) - \overline{W}(D_{a,ij}^{rr}) \right), \quad (27)$$

where $D_{a,ij}^{\times}$ (resp. $D_{a,ij}^{\parallel}$) is the diagram obtained by adding to $D_{a,ij}$ two new chords: the chord connecting the left end of b_i with the right end of b_j and the chord connecting the left end of b_j with the right end of b_i (resp. the cord connecting the left ends of b_i and b_j and the chord connecting their right ends) and $D_{a,ij}^{lr}$ (resp. $D_{a,ij}^{rl}$, $D_{a,ij}^{ll}$, and $D_{a,ij}^{rr}$) is the diagram obtained by adding to $D_{a,ij}$ a new chord connecting the left end of b_i and the right end of b_j (resp. the right end of b_i and the left end of b_j ; the left ends of b_i and b_j ; the right ends of b_i and b_j) assuming that the chord a is drawn vertically. (See Fig. 2 where a is the vertical chord and b_i, b_j are chords which intersect a so that b_i is the upper chord and b_j is the lower chord.)

²In fact, the identity (24) goes back to Euler's theory of the motion of a rigid body. In the language of 20th century physics it looks like $\epsilon_{\alpha\beta\gamma}\epsilon_{\rho\gamma\sigma} = \delta_{\alpha\sigma}\delta_{\beta\rho} - \delta_{\alpha\rho}\delta_{\beta\sigma}$, where δ is the Kronecker delta, and ϵ is the standard completely antisymmetric tensor in \mathbf{R}^3 .

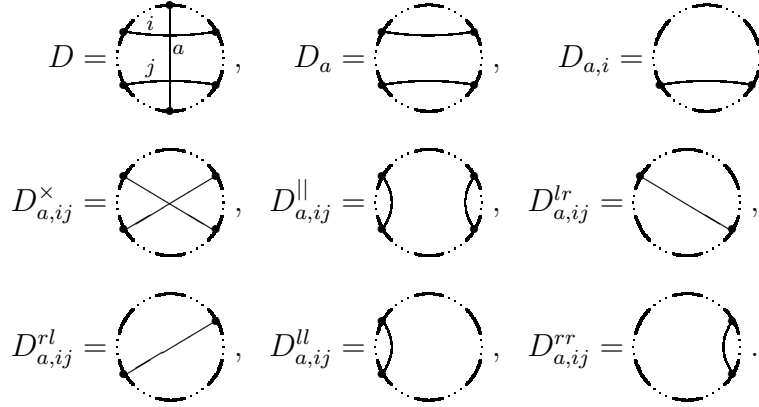


Figure 2: Chord diagrams entering the recursion formulas

Since all the diagrams in the right-hand side of (26) and (27) have fewer chords than D , this allows us to compute the value of the weight systems W and \overline{W} on any chord diagram recursively.

Proof. Relation (i) for the framed invariant was proved by Chmutov and Varchenko in [5]. It follows from Lagrange's identity (24) by induction on the number of chords in I_a .

Let us prove relation (ii) for the unframed invariant.

By the deframing formula (15) we have

$$\begin{aligned}
 \overline{W}(D) &= (U \cdot W)(D) = \sum_{E \subset D} U(D \setminus E) \cdot W(E) = \sum_{E \subset D} (-c)^{|D \setminus E|} W(E) \\
 &= \sum_{E \not\ni a} (-c)^{|D \setminus E|} W(E) + \sum_{E \ni a} (-c)^{|D \setminus E|} W(E) \\
 &= \sum_{E \subset D_a} \left((-c)^{|D \setminus E|} W(E) + (-c)^{|D \setminus E|-1} W(E + a) \right)
 \end{aligned} \tag{28}$$

Applying the recursion formula (i) to $W(E + a)$ in (28), we have

$$\begin{aligned}
 \overline{W}(D) &= \sum_{E \subset D_a} \left((-c)^{|D \setminus E|} W(E) + (-c)^{|D \setminus E|-1} \left((c - 2|I_a \cap E|) W(E) \right. \right. \\
 &\quad \left. \left. + \sum_{\substack{i < j, \\ b_i, b_j \in E}} 2(W(E_{ij}^{||}) - W(E_{ij}^{\times})) \right) \right) \\
 &= \sum_{E \subset D_a} -2(-c)^{|D \setminus E|-1} \left(|I_a \cap E| \cdot W(E) - \sum_{\substack{i < j, \\ b_i, b_j \in E}} (W(E_{ij}^{||}) - W(E_{ij}^{\times})) \right) \\
 &= \sum_{E \subset D_a} \sum_{\substack{i \\ b_i \in E}} -2(-c)^{|D \setminus E|-1} \cdot W(E)
 \end{aligned} \tag{29}$$

$$+ \sum_{E \subset D_a} \sum_{\substack{i < j, \\ b_i, b_j \in E}} 2(-c)^{|D \setminus E| - 1} \left(W(E_{ij}^{\parallel}) - W(E_{ij}^{\times}) \right). \quad (30)$$

Changing the order of summation in (29) and (30) we get

$$\begin{aligned} \overline{W}(D) &= \sum_i \sum_{\substack{E \subset D_a \\ E \ni b_i}} -2(-c)^{|D \setminus E| - 1} W(E) \\ &\quad + \sum_{i < j} \sum_{\substack{E \subset D_a \\ E \ni b_i, b_j}} 2(-c)^{|D \setminus E| - 1} \left(W(E_{ij}^{\parallel}) - W(E_{ij}^{\times}) \right) \end{aligned} \quad (31)$$

$$= \sum_i -2S_i + \sum_{i < j} 2(S_{ij}^{\parallel} - S_{ij}^{\times}), \quad (32)$$

where

$$\begin{aligned} S_i &= \sum_{\substack{E \subset D_a \\ E \ni b_i}} (-c)^{|D \setminus E| - 1} W(E) \\ &= \sum_{E \subset D_a} (-c)^{|D \setminus E| - 1} W(E) - \sum_{E \subset D_{a,i}} (-c)^{|D \setminus E| - 1} W(E) \\ &= \sum_{E \subset D_a} U(D_a \setminus E) W(E) + \sum_{E \subset D_{a,i}} cU(D_{a,i} \setminus E) W(E), \end{aligned}$$

and by the deframing formula (15) we obtain

$$S_i = \overline{W}(D_a) + c\overline{W}(D_{a,i}). \quad (33)$$

For the term S_{ij}^{\parallel} in (32) we have (by the inclusion-exclusion principle and the deframing formula)

$$\begin{aligned} S_{ij}^{\parallel} &= \sum_{\substack{E \subset D_a \\ E \ni b_i, b_j}} (-c)^{|D \setminus E| - 1} W(E_{ij}^{\parallel}) \\ &= \sum_{E \subset D_{a,ij}^{\parallel}} (-c)^{|D \setminus E| - 1} W(E) - \sum_{E \subset D_{a,ij}^{\parallel}} (-c)^{|D \setminus E| - 1} W(E) \\ &\quad - \sum_{E \subset D_{a,ij}^{rr}} (-c)^{|D \setminus E| - 1} W(E) + \sum_{E \subset D_{a,ij}} (-c)^{|D \setminus E| - 1} W(E) \\ &= \sum_{E \subset D_{a,ij}^{\parallel}} U(D_{a,ij}^{\parallel} \setminus E) W(E) + \sum_{E \subset D_{a,ij}^{\parallel}} cU(D_{a,ij}^{\parallel} \setminus E) W(E) \\ &\quad + \sum_{E \subset D_{a,ij}^{rr}} cU(D_{a,ij}^{rr} \setminus E) W(E) + \sum_{E \subset D_{a,ij}} c^2U(D_{a,ij} \setminus E) W(E) \\ &= \overline{W}(D_{a,ij}^{\parallel}) + c\overline{W}(D_{a,ij}^{\parallel}) + c\overline{W}(D_{a,ij}^{rr}) + c^2\overline{W}(D_{a,ij}). \end{aligned} \quad (34)$$

Similarly, for S_{ij}^\times in (32) we get

$$\begin{aligned} S_{ij}^\times &= \sum_{\substack{E \subset D_a \\ E \ni b_i, b_j}} (-c)^{|D \setminus E| - 1} W(E_{ij}^\times) \\ &= \overline{W}(D_{a,ij}^\times) - c \overline{W}(D_{a,ij}^{ll}) - c \overline{W}(D_{a,ij}^{rr}) - c^2 \overline{W}(D_{a,ij}) \end{aligned} \quad (35)$$

Substituting (33), (34), and (35) in (32), we obtain the right hand side of (ii). \square

Corollary 3.2 *Let $\overline{w}_{j,n}$ be the order n coefficients of the universal weight system*

$$\overline{W}_{sl_2} = \sum_{j,n} \overline{w}_{j,n} c^j$$

and

$$\overline{W}_k = \sum_{n \geq 0} \overline{w}_{2n-k,n} \in \mathcal{W} . \quad (36)$$

Then

(i) *the Vassiliev series (36) vanishes for $k < 0$, i.e.*

$$\overline{w}_{j,n} = 0 \text{ for } j > n/2 , \quad (37)$$

(ii) *the leading term series $\overline{W}_0 = \sum_{n \geq 0} \overline{w}_{2n,n}$ satisfies the recursion relation*

$$\overline{W}_0(D) = -2 \sum_i \overline{W}_0(D_{a,i}) - 2 \sum_{i < j} \left(\overline{W}_0(D_{a,ij}^{lr}) + \overline{W}_0(D_{a,ij}^{rl}) - \overline{W}_0(D_{a,ij}^{ll}) - \overline{W}_0(D_{a,ij}^{rr}) \right) , \quad (38)$$

(iii) *the lower order series \overline{W}_k satisfies the recursion relation*

$$\begin{aligned} \overline{W}_k(D) &= -2 \sum_i \overline{W}_k(D_{a,i}) - 2 \sum_{i < j} \left(\overline{W}_k(D_{a,ij}^{lr}) + \overline{W}_k(D_{a,ij}^{rl}) - \overline{W}_k(D_{a,ij}^{ll}) - \overline{W}_k(D_{a,ij}^{rr}) \right) \\ &\quad - 2|I_a| \overline{W}_{k-1}(D_a) + 2 \sum_{i < j} \left(\overline{W}_{k-1}(D_{a,ij}^{ll}) - \overline{W}_{k-1}(D_{a,ij}^\times) \right) . \end{aligned} \quad (39)$$

where the diagrams D_a , $D_{a,i}$, etc. are the same as in Theorem 3.1 (see Fig. 2).

Proof. Relation (iii) follows immediately from (27) and (36). Now (ii) and (i) follow from (iii) by induction on the number of chords in D , since $\overline{W}_k(\Theta) = 0$ for all k . \square

We will show that relations (37) and (38) imply the Melvin-Morton conjecture (Theorem 1.1), but first we have to recall some facts on the Alexander weight system and its relation with the universal $gl(1|1)$ invariant.

4 Alexander and $gl(1|1)$ universal weight system

The universal weight system for the Lie superalgebra $gl(1|1)$ was studied in [6]. Here we recall the main results of [6], and show that together with Corollary 3.2 the recursion relation for $W_{gl(1|1)}$ implies the Melvin-Morton conjecture.

4.1 The $gl(1|1)$ weight system

Let $V \cong \mathbf{C}^{1|1}$ be a $(1|1)$ -dimensional superspace. The Lie superalgebra of endomorphisms of V is called $gl(1|1)$.

The bilinear form

$$\langle x, y \rangle_{str} = str(xy)$$

on $gl(1|1)$ is invariant and non-degenerate. (The *supertrace* of a $(m|n) \times (m|n)$ matrix $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ is defined as $str M = tr A - tr B$). Therefore, we can consider the universal weight system $W_{gl(1|1)}$ with values in $Z(\mathcal{U}(gl(1|1))) = \mathbf{C}[h, c]$, where $h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in gl(1|1)$ and c is the quadratic Casimir.

We have the following analogs of the fundamental sl_2 relations (23) and (24) between invariant tensors on $L = gl(1|1)$ defined by the diagrams B and K (see (18)).

$$W_L \left(\text{diagram of a circle on a line} \right) = -2h^2. \quad (40)$$

and

$$W_L(K) = \frac{1}{2}W_L(M), \quad (41)$$

where

$$M = \text{diagram 1} + \text{diagram 2} - \text{diagram 3} - \text{diagram 4}.$$

Relation (40) shows in particular that the Killing form for $gl(1|1)$ is degenerate. The fundamental relation (41) was found in [6], where we derived the following recursive formulas for the universal $gl(1|1)$ weight systems.

Theorem 4.1 *Let W be the universal $gl(1|1)$ weight system. In the notations of Theorem 3.1 we have*

(i) $W(D)$ is a polynomial in c and h^2 satisfying

$$\begin{aligned} W(D) = & cW(D_a) + h^2 \sum_i W(D_{a,i}) \\ & - h^2 \sum_{i < j} \left(W(D_{a,ij}^{lr}) + W(D_{a,ij}^{rl}) - W(D_{a,ij}^{ll}) - W(D_{a,ij}^{rr}) \right), \end{aligned} \quad (42)$$

and

(ii) the deframed $gl(1|1)$ weight system \overline{W} is given by setting $c = 0$ in W and satisfies the following recursion relation.

$$\overline{W}(D) = h^2 \sum_i \overline{W}(D_{a,i}) - h^2 \sum_{i < j} \left(\overline{W}(D_{a,ij}^{lr}) + \overline{W}(D_{a,ij}^{rl}) - \overline{W}(D_{a,ij}^{ll}) - \overline{W}(D_{a,ij}^{rr}) \right). \quad (43)$$

The following result connects the Alexander polynomial with the $gl(1|1)$ universal invariant.

Proposition 4.2 ([6]) *The deframed $gl(1|1)$ weight system coincides with the Vassiliev series of the renormalized Alexander invariant $\hat{\Delta}$ (19).*

4.2 Melvin-Morton expansion

Information on the universal sl_2 and $gl(1|1)$ weight systems allows us to study the canonical Vassiliev invariants appearing in the Melvin-Morton expansion (3).

Proposition 4.3 *The Vassiliev series $W_0 = \sum_{n \geq 0} w_{nn}$ of the diagonal coefficients v_{nn} of the Melvin-Morton expansion (5) satisfies the recursion relation*

$$W_0(D) = - \sum_i W_0(D_{a,i}) - \sum_{i < j} \left(W_0(D_{a,ij}^{lr}) + W_0(D_{a,ij}^{rl}) - W_0(D_{a,ij}^{ll}) - W_0(D_{a,ij}^{rr}) \right). \quad (44)$$

Proof. The quadratic Casimir $c \in Z(\mathcal{U}(sl_2))$ in the d -dimensional irreducible representation acts as a multiplication by $(d^2 - 1)/2$. Therefore, the weight system w_{nn} can be expressed via the coefficients \overline{w}_{jn} of (36) as

$$w_{nn} = \begin{cases} 2^{-n} \overline{w}_{n,n/2}, & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

and the relation (44) now follows from (38). \square

Comparing the relations (44) and (43) we come to a new proof of Proposition 2.2 and, therefore, of the Melvin-Morton conjecture.

Proof of Proposition 2.2.

Part (i) follows from Corollary 3.2.(i), since $\widehat{W}_{sl}(d) = \overline{W}_{sl_2}(\frac{d^2-1}{2})$ and the coefficients w_{in} in (20) with $i > n$ are combinations of the coefficients \overline{w}_{jn} in (36) with $j > n/2$ and therefore vanish.

We will show that (22) follows from recursion relations for W_0 (44) and for C (43) by induction.

By the product formula (12) and relations (44) and (43) we have

$$\begin{aligned}
(W_0 \cdot C)(D) &= \sum_{E \subset D} W_0(D \setminus E)C(E) \\
&= \sum_{E \subset D_a} (W_0(D \setminus E)C(E) + W_0(D_a \setminus E)C(E + a)) \\
&= \sum_{E \subset D_a} \left[\sum_{\substack{i \\ b_i \notin E}} -W_0(D_{a,i} \setminus E)C(E) + \sum_{\substack{i \\ b_i \in E}} W_0(D_a \setminus E)C(E_i) \right. \\
&\quad - \sum_{\substack{i < j \\ b_i, b_j \notin E}} \left(W_0(D_{a,ij}^{lr} \setminus E) + W_0(D_{a,ij}^{rl} \setminus E) - W_0(D_{a,ij}^{ll} \setminus E) - W_0(D_{a,ij}^{rr} \setminus E) \right) C(E) \\
&\quad \left. - \sum_{\substack{i < j \\ b_i, b_j \in E}} W_0(D_a \setminus E) \left(C(E_{ij}^{lr}) + C(E_{ij}^{rl}) - C(E_{ij}^{ll}) - C(E_{ij}^{rr}) \right) \right] \\
&= \sum_i \left(\sum_{\substack{E \subset D_a \\ E \not\ni b_i}} -W_0(D_{a,i} \setminus E)C(E) + \sum_{\substack{E \subset D_a \\ E \ni b_i}} W_0(D_a \setminus E)C(E_i) \right) \\
&\quad - \sum_{i < j} \left(\left(\sum_{\substack{E \subset D_a \\ E \not\ni b_i, b_j}} W_0(D_{a,ij}^{lr} \setminus E)C(E) + \sum_{\substack{E \subset D_a \\ E \ni b_i, b_j}} W_0(D_a \setminus E)C(E_{ij}^{lr}) \right) \right. \\
&\quad \left. + \left(\sum_{\substack{E \subset D_a \\ E \not\ni b_i, b_j}} W_0(D_{a,ij}^{rl} \setminus E)C(E) + \sum_{\substack{E \subset D_a \\ E \ni b_i, b_j}} W_0(D_a \setminus E)C(E_{ij}^{rl}) \right) \right. \\
&\quad \left. - \left(\sum_{\substack{E \subset D_a \\ E \not\ni b_i, b_j}} W_0(D_{a,ij}^{ll} \setminus E)C(E) + \sum_{\substack{E \subset D_a \\ E \ni b_i, b_j}} W_0(D_a \setminus E)C(E_{ij}^{ll}) \right) \right. \\
&\quad \left. - \left(\sum_{\substack{E \subset D_a \\ E \not\ni b_i, b_j}} W_0(D_{a,ij}^{rr} \setminus E)C(E) + \sum_{\substack{E \subset D_a \\ E \ni b_i, b_j}} W_0(D_a \setminus E)C(E_{ij}^{rr}) \right) \right) \\
&= \sum_i \left(-(W_0 \cdot C)(D_{a,i}) + (W_0 \cdot C)(D_{a,i}) \right) \\
&\quad - \sum_{i < j} \left((W_0 \cdot C)(D_{a,ij}^{lr}) + (W_0 \cdot C)(D_{a,ij}^{rl}) - (W_0 \cdot C)(D_{a,ij}^{ll}) - (W_0 \cdot C)(D_{a,ij}^{rr}) \right) \\
&= \sum_{i < j} \left((W_0 \cdot C)(D_{a,ij}^{ll}) + (W_0 \cdot C)(D_{a,ij}^{rr}) - (W_0 \cdot C)(D_{a,ij}^{lr}) - (W_0 \cdot C)(D_{a,ij}^{rl}) \right) .
\end{aligned}$$

Now, since $(W_0 \cdot C)(D) = 1$ for $|D| = 0$ and $(W_0 \cdot C)(D) = 0$ for $|D| = 1$ we see by induction on $|D|$ that $(W_0 \cdot C)(D) = 0$ if $|D| \geq 1$. \square

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